Exactly solvable scale-free network model

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We study a deterministic scale-free network recently proposed by Barabási, Ravasz, and Vicsek. We find that there are two types of nodes: the hub and rim nodes, which form a bipartite structure of the network. We first derive the exact numbers $P(k)$ of nodes with degree k for the hub and rim nodes in each generation of the network, respectively. Using this, we obtain the exact exponents of the distribution function $P(k)$ of nodes with k degree in the asymptotic limit of $k\to\infty$. We show that the degree distribution for the hub nodes exhibits the scale-free nature, $P(k)\propto k^{-\gamma}$ with $\gamma=\ln 3/\ln 2=1.584\,962$, while the degree distribution for the rim nodes is given by $P(k)\propto e^{-\gamma' k}$ with $\gamma'=\ln(3/2)=0.405\,465$. Second, we analytically calculate the second-order average degree of nodes, $\tilde{d}$. Third, we numerically as well as analytically calculate the spectra of the adjacency matrix A for representing topology of the network. We also analytically obtain the exact number of degeneracies at each eigenvalue in the network. The density of states (i.e., the distribution function of eigenvalues) exhibits the fractal nature with respect to the degeneracy. Fourth, we study the mathematical structure of the determinant of the eigenequation for the adjacency matrix. Fifth, we study hidden symmetry, zero modes, and its index theorem in the deterministic scale-free network. Finally, we study the nature of the maximum eigenvalue in the spectrum of the deterministic scale-free network. We will prove several theorems for it, using some mathematical theorems. Thus, we show that most of all important quantities in the network theory can be analytically obtained in the deterministic scale-free network model of Barabási, Ravasz, and Vicsek. Therefore, we may call this network model the exactly solvable scale-free network.

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I. INTRODUCTION

There has been notable progress in the study of the so-called scale-free network (SFN) [1–10] in recent years. In the network theory, the random network model was first invented by Erdös and Rényi [11]. Recently it was generalized to small-world network models [12–20]. Furthermore, about five years ago, the SFN was discovered by studying the network geometry of the internet [1–3,21–24]. The Faloutsos brothers [1] and Albert, Jeong, and Barabási [2,3,21–24] first showed the scale-free nature of the internet geometry and opened up an area for studying very complex and growing network systems such as the internet, biological evolution, metabolic reaction, epidemic disease, human sexual relationship, and economy. These are nicely summarized in the reviews of Barabási [2].

As was studied in the literature [2], the nature of these SFN's is characterized by a power-law behavior of the distribution function. Here the number of nodes with order k can be fit by $P(k)\propto k^{-\gamma}$, where $\gamma\approx 1-4$. In order to show the power-law distribution of the SFN, Albert and Barabási first proposed a very simple model called the Albert-Barabási (AB) SFN model [2,3,21–24].

This system is constructed by the following process: Initially we put m_0 nodes as seeds for the system. Every time a new node is added, m new links are connected from the node to the existing nodes in the system with a preferential attachment probability $\Pi_i(k_i)=k_i/\sum_{i=1}^{N-1}k_i$, where k_i is the number of links at the ith node. The development of this model is described by a continuum model $dk_i/d\tau=m\Pi_i(k_i)=mk_i/2\tau$. Then at time τ the system consists of $N(\tau)$ nodes and $L(\tau)$ links with $L(\tau)=\frac{1}{2}\sum_{i=1}^{N(\tau)}k_i$. As studied by Barabási and coworkers [2] this model exhibits an exact exponent of $\gamma=3$ for the power law. Thus, it has been concluded that the essential points of why a network grows to a SFN are attributed to the growth of the system and the preferential attachment of new nodes to old nodes existing already in the network.

From the above context, the time evolution to construct a SFN has been intensively studied in the AB model as well as other models. And many works have appeared regarding nodes and links as metaphysical objects such as agents and relationships in an area of science [2]. However, most approaches were based on a numerical approach. And the spectra of the adjacency matrix A for the SFN have been studied numerically [1,2,25–27]. Therefore, apart from the purposes for the numerical analysis, the continuous-time SFN models such as the AB model are not good enough at the microscopic level to see what is going on in the network geometry.

Instead of such a continuous-time SFN model, there has been proposed a new type of the SFN models, sometimes called deterministic models [28] and hierarchical SFN models [29] (we would like to call it the DSFN in this paper). In the former, the study showed a power-law behavior of the network analytically, while in the latter, the study showed it numerically. However, we do not know much about the properties of these models yet.

On the other hand, there is a very important problem on the maximum eigenvalue λ_{max} of the adjacency matrix A. As

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was numerically studied [25–27], the maximum eigenvalue for the AB model is bounded by $\sqrt{k_{max}}$ such that $\lambda_{max} \leqslant \text{const} \times \sqrt{k_{max}}$, where k_{max} means the maximum order of nodes. And numerical studies showed that $k_{max} \propto \sqrt{N}$. And therefore, the numerical studies validated $\lambda_{max} \leqslant \text{const} \times N^{1/4}$. On this problem, very recently, Chung, Lu, and Vu [31] have proved a very general theorem: In a complex network model, λ_{max} is always bounded by the lower and upper bounds such that $a \leqslant \lambda_{max} \leqslant b$. Define the second-order average degree of nodes, $\tilde{d}$ (see Sec. IV). Then, (C1) if the exponent $\gamma > 2.5$, then $\text{const} \times \tilde{d} \leqslant \lambda_{max} \leqslant \text{const} \times \sqrt{k_{max}}$. (C2) If the exponent $\gamma < 2.5$, then $\text{const} \times \sqrt{k_{max}} \leqslant \lambda_{max} \leqslant \text{const} \times \tilde{d}$. (C3) And if the exponent $\gamma = 2.5$, then a transition happens. From this theorem, the AB model belongs to the first category since $\gamma = 3$. However, in spite of the seemingly important theorem, because of the lack of other good examples other than the AB model, examples of the other categories have not yet been so well known so far.

So the purpose of this paper is to study in much detail the DSFN model proposed by Barabási, Ravasz, and Vicsek [28] in order to give a good example of the other category of the theorem. This study will provide a rigorous treatment of the complex network model.

The organization of the paper is the following. In Sec. II, we will introduce the DSFN model that was first studied by Barabási, Ravasz, and Vicsek [28]. In Sec. III, we will present the exact numbers of nodes and degrees in the network and calculate the exact scaling behavior of the nodes. In Sec. IV, we will calculate the exact number of the second-order average degree $\tilde{d}$, using the exact number distributions of the nodes and degrees. In Sec. V, we will introduce our formalism of the paper such as the so-called adjacency matrix in the network theory and the eigenvalue problem of the system. In Sec. VI, we will present the numerical results of the spectra of the adjacency matrices for the DSFN's up to the $n = 7$ generation. And we will derive the exact numbers of degeneracies in the spectrum. In Sec. VII, we will present an analytical method that deduces the exact numbers of degeneracies in terms of irreducible polynomials for the DSFN. We will also present some conjectures on the polynomials and discuss the role of the roots of the irreducible polynomials. In Sec. VIII, we show that there is a hidden symmetry in the adjacency matrix of the DSFN. And we will discuss the zero modes and its index theorem in the DSFN. In Sec. IX, we will discuss the nature of the maximum eigenvalue in the spectrum of the DSFN. We will prove several theorems using some mathematical theorems such as the Perron-Frobenius theorem. And we will discuss the relationship between the result of our theory and that of the theorem of Chung, Lu, and Vu [31]. In Sec. X, conclusions will be made.

II. DETERMINISTIC SCALE-FREE NETWORK

Let us introduce the DSFN model invented by Barabási, Ravasz, and Vicsek [28]. The development of this network is shown in Fig. 1. The black and white nodes show the hub and rim nodes. We call the most connected hub and rim the root and leaf, respectively.

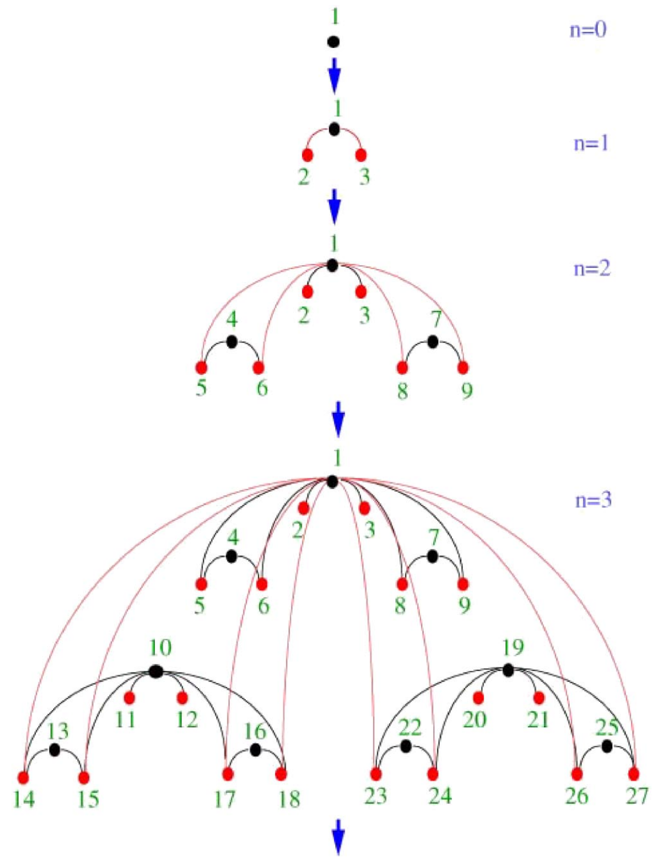

FIG. 1. (Color online) The deterministic scale-free network. The black and white nodes show the hub and rim nodes. We call the most connected hub and rim the root and leaf, respectively. This network is a bipartite structure.

In this model the total number of nodes, $N(n)$, the total number of links, $L(n)$, and the maximum number of links, $k_{max}(n)$, are given by

$$N(n) = 3^n = 0,1,2,3,4, \dots , \tag{1}$$

$$L(n) = 2(3^n - 2^n) = 0,2,10,38,130, \dots , \tag{2}$$

$$k_{max}(n) = 2(2^n - 1) = 0,2,6,14,30, \dots , \tag{3}$$

respectively.

Let us consider the average link number (i.e., the average degree) $\langle k \rangle$ of a network. It is defined by

$$\langle k \rangle(n) \equiv \frac{1}{N(n)} \sum_{i=1}^{N(n)} k_i . \tag{4}$$

The meaning of this is just the number of links per node (i.e., the average degree). We may call it the first-order average degree. On the other hand, if we use the number $P(k)$ of nodes with k degree, then we can write the average as

$$\langle k \rangle(n) = \frac{1}{N(n)} \sum_{k=1}^{k_{max}(n)} kP(k). \tag{5}$$

Therefore, we find that the conversion is carried out by

$$P(k) = \sum_{i=1}^{N(n)} \delta_{k,k_i}. \tag{6}$$

Now, we are able to calculate $\langle k \rangle$ for our DSFN. Substituting Eqs. (1) and (2) into Eq. (4), we obtain

$$\langle k \rangle (n) = \frac{2L(n)}{N(n)} = 4\left[1 - \left(\frac{2}{3}\right)^n\right] \xrightarrow{n\to\infty} 4. \tag{7}$$

In this way, even though the network becomes very complex as $n \to \infty$, the average approaches a finite constant 4. This is due to the following fact: In this DSFN as the iteration is repeated, the order of the most connected hub becomes large indefinitely while its number remains very few [i.e., $\approx o(1)$]. On the other hand, the numbers of the very few connected nodes become large indefinitely. Hence the sum of the magnitude of order (k) times the number $[P(k)]$ of nodes with the order k may remain finite.

III. EXACT NUMBERS OF NODES AND DEGREES

Let us find the exact numbers of nodes and degrees. This would be very crucial for our later purposes in order to evaluate many quantities in the network theory.

As was discussed by Barabási, Ravasz, and Vicsek [28], in the DSFN there are two categories of nodes called the "hub" nodes and "rim" nodes. As they called the most connected hub node the "root" (shown as black dots in Fig. 1), we would like to call the most connected rim node the "leaf" (shown as white dots in Fig. 1). From seeing Fig. 1, the locations of the root node and the leaf nodes look very similar to those of a hub and rims in an umbrella. While there exists only one roof node in each generation of the network, the number of leaves can increase very rapidly.

Let us first consider the hub nodes. In the ith step, the degree of the root is $2^{i+1}-2$. In the next iteration two copies of this hub will appear in the two newly added units. As we iterate further, in the nth step 3^{n-i} copies of this hub will be present in the network. However, the two newly created copies will not increase their degree after further iterations. Therefore, after n iterations there are $2\times 3^{n-i-1}$ nodes with degree $2^{i+1}-2$.

Let us next consider the rim nodes. In the ith step, the degree of the most connected rim, the leaf, is i. And the number of such nodes is 2^i. In the next iteration one copy of the leaves will be kept the same and two copies of the leaves will appear in the two newly added. As we iterate further, in the nth step 3^{n-i} copies of the leaves will be present in the network. Therefore, after n iterations there are $2^i \times 3^{n-i-1}$ nodes with degree i.

Now, denote by k the degree of nodes and denote by $P(k)$ the total number of nodes with degree k. Hence we get Table I.

As was shown by Barabási, Ravasz, and Vicsek [28], consideration of the root nodes is enough to derive the scaling exponent of the distribution function $P(k)$ for the root nodes in the network. Picking up the $2\times 3^{n-1-i}$ nodes with degree $2^{i+1}-2$, we can regard $P(k)$ as $2\times 3^{n-1-i}$ and k as $2^{i+1}-2$. Then eliminating i, we can derive $P(k) \propto k^{-\gamma}$, where $\gamma = \ln 3/\ln 2 = 1.584\,962$. This shows a scale-free nature (i.e., the fractal nature) of the hub nodes in the network as we expect [30].

TABLE I. The number $P(k)$ of nodes with degree k for the root nodes and leaf nodes.

Root nodes		Leaf nodes	
k	$P(k)$	k	$P(k)$
2	$2\times 3^{n-2}$	1	$2\times 3^{n-2}$
6	$2\times 3^{n-3}$	2	$2^2\times 3^{n-3}$
14	$2\times 3^{n-4}$	3	$2^3\times 3^{n-4}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2^{i+1}-2$	$2\times 3^{n-1-i}$	i	$2^i\times 3^{n-1-i}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2^{n-1}-2$	2×3	$n-2$	$2^{n-2}\times 3$
2^n-2	2	$n-1$	$2^{n-1}\times 1$
$2^{n+1}-2$	1	n	2^n

On the other hand, it is not true for the leaves in the network. In this case, regarding $P(k)$ as $2^i\times 3^{n-1-i}$ and k as i, we find $P(k) \propto \left(\frac{2}{3}\right)^k = e^{-\gamma' k}$, where $\gamma' = \ln\left(\frac{3}{2}\right) = 0.405\,465$. This shows that the scaling nature of the leaf nodes is not scale free but exponential.

In this way, the scaling nature of the roots and that of the leaves in the DSFN are different from each other. Thus, we are led to a certain model which consists of a *multifractal* nature of the complex networks.

IV. SECOND-ORDER AVERAGE DEGREE

Let us calculate the second-order average degree $\tilde{d}(n)$. It is defined by

$$\tilde{d}(n) \equiv \frac{1}{L(n)} \sum_{i=1}^{N(n)} k_i^2 = \frac{1}{L(n)} \sum_{k=1}^{k_{max}(n)} k^2 P(k). \tag{8}$$

This quantity was recently introduced by Chung, Lu, and Vu [31]. Roughly speaking, the meaning of this quality is the average degree per link. In other words, it is the average degree weighted with the preferential attachment such that

$$\tilde{d}(n) \equiv \sum_{i=1}^{N(n)} k_i \Pi_i(k_i). \tag{9}$$

We derive

$$\tilde{d}(n) = 2\frac{\langle k^2 \rangle (n)}{\langle k \rangle (n)}, \tag{10}$$

where $\langle k^2 \rangle (n)$ is defined by

$$\langle k^2 \rangle (n) = \frac{1}{N(n)} \sum_{i=1}^{N(n)} k_i^2 = \frac{1}{N(n)} \sum_{k=1}^{k_{max}(n)} k^2 P(k), \tag{11}$$

the second moment per node. As was shown before, the average degree $\langle k \rangle (n)$ converges to 4 as $n \to \infty$. Hence, the second-order average degree $\tilde{d}(n)$ becomes proportional to the second moment $\langle k^2\rangle(n)$ in the limit such that

$$\tilde{d} = \frac{\langle k^2\rangle(n)}{2}. \tag{12}$$

Before going to calculate the second-order average degree, let us first check whether or not the distributions given in the tables reproduce the correct results for the total numbers of nodes and links. This problem is a trivial one. However, as we will see, this is very instructive for our purpose here in order to see what is going on in the problem.

Let us show that the distributions of the nodes and degrees are exact. We can sum them up as follows: $\Sigma_{k\in all}P(k)=\Sigma_{k\in root}P(k)+\Sigma_{k\in leaf}P(k)=3^{n-1}+2\times 3^{n-1}=3^n=N(n)$. This proves Eq. (1).

Let us next consider the total number of links in the network. We calculate it for the root and leaf nodes, separately, as $\Sigma_{k\in root}kP(k)=2(3^n-2^n)$, $\Sigma_{k\in leaf}kP(k)=2(3^n-2^n)$. In this way, explicitly using the exact numbers of nodes and degrees, we can show that each sum produces the total number of links in the network as we have expected. Hence, this proves Eq. (2).

This situation encourages us to perform the calculation of the second-order average degree of Eq. (8). Let us do this next. In Eq. (8), we need to separate it into two parts of the sum as follows:

$$\tilde{d}(n) = \frac{1}{L(n)}\left(\sum_{k\in root} k^2P(k) + \sum_{k\in leaf} k^2P(k)\right). \tag{13}$$

As a result, we can easily calculate the second-order average degree as follows:

$$\tilde{d}(n) = \frac{1}{1-\left(\frac{2}{3}\right)^n}\left[3\left(\frac{4}{3}\right)^n - \left(n+\frac{1}{2}\right)\left(\frac{2}{3}\right)^n - \frac{5}{2}\right] \overset{n\to\infty}{\longrightarrow} 3\left(\frac{4}{3}\right)^n \to \infty. \tag{14}$$

In this way, the second-order average degree $\tilde{d}$ has been calculated explicitly, and we have shown that it diverges as an exponential law.

V. FORMALISM

A. Adjacency matrix

Let us consider the adjacency matrix A in the theory of networks [1,2,28,29,31]. This matrix is very important for the theory of networks, since it represents the topology of the network structure. Denote by a_{ij} a component corresponding to a link between the ith node and the jth node. In the theory of networks, the elements have only 0 or 1 according to whether or not there is a link. Therefore, in general, the adjacency matrix is defined by

$$a_{ij} = \begin{cases} 1 & \text{if there is a link,} \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

Let us consider the adjacency matrix for the DSFN. Denote by A_n the adjacency matrix for the nth generation of the network. Using the numbering given in Fig. 1, the adjacency matrix is defined by

$$A_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{16}$$

$$A_2 = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & & & & & & \\ 1 & 0 & 0 & & & & & & \\ 0 & & & 0 & 1 & 1 & & & \\ 1 & & & 1 & 0 & 0 & & & \\ 1 & & & 1 & 0 & 0 & & & \\ 0 & & & & & & 0 & 1 & 1 \\ 1 & & & & & & 1 & 0 & 0 \\ 1 & & & & & & 1 & 0 & 0 \end{pmatrix}, \tag{17}$$

and so forth. Here all other blanks stand for zeros. We omit them just for seeing the sparse matrix structure corresponding to the network geometry of the DSFN. What is important here in the above is that the adjacency matrix for a certain generation of the network can be almost three-block diagonal by those for the last generation of the network. From this, the fractal nature of the adjacency matrices for the DSFN is now very clear.

B. Eigenequations

Let us next consider the eigenvalue problem for the adjacency matrix in the DSFN. The eigenequation for the nth generation of the network is given by

$$A_n\vec{X}_n = \lambda\vec{X}_n, \quad \vec{X}_n^t\cdot\vec{X}_n = 1, \tag{18}$$

where A_n is the $3^n\times 3^n$ matrix and $\vec{x}_n$ the 3^n-dimensional vector with its transpose $\vec{x}_n^t$. This reduces to the following eigenvalue problem:

$$\det[\lambda I_n - A_n] = 0, \tag{19}$$

where I_n stands for the $3^n\times 3^n$ unit matrix. Denote by $D_n(\lambda)$ the determinant on the left-hand side. This can be formally expanded with respect to λ as

$$D_n(\lambda) = \lambda^{3^n} - a_1\lambda^{3^n-1} + a_2\lambda^{3^n-2} + \cdots + (-1)^{3^n-1}a_{3^n-1}\lambda + (-1)^{3^n}a_{3^n}. \tag{20}$$

We note the following: Since the λ is an eigenvalue of the adjacency matrix A_n, from knowledge of linear algebra, we are able to derive the equation $D_n(A_n)=0$. As we will see later, all the terms of even number powers of λ vanish in the DSFN. This is attributed to the topology of the network.

VI. NUMERICAL CALCULATION OF THE SPECTRA

Let us obtain the spectra of the adjacency matrices, discussed in the previous section. We use a computer calcula-

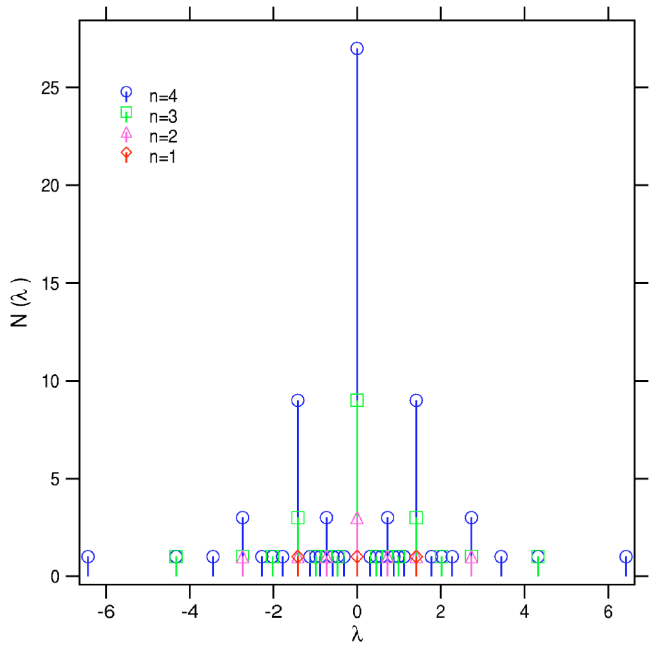

FIG. 2. (Color online) The spectrum of adjacency matrix for the deterministic scale-free network. The spectrum of adjacency matrix for the network of the nth generation is shown for $n=1$ (triangles), $n=2$ (circles), $n=3$ (squares), and $n=4$ (diamonds), respectively.

tion for this purpose. The calculated spectra for the adjacency matrix for each generation of the network are shown in Fig. 2.

From the numerical results, we find the following very important characters of the spectra: (i) The maximum eigenvalue $\lambda_1^{(n)}$ at the nth generation of the network becomes the second largest eigenvalue $\lambda_2^{(n+1)}$ at the $(n+1)$th generation of the network.

(ii) Similarly for the other eigenvalues, all the eigenvalues appearing at the previous generations of network always exist in the eigenvalues appearing at the new generation of the network.

(iii) *The spectrum consists of highly degenerate levels*. For example, for $n=1$ there are three levels (shown by triangles) of $\lambda=0, \pm\sqrt{2}$ with single degeneracy. For $n=2$ there are nine levels (shown by circles) in the spectrum. There is only one peak at the center level of $\lambda=0$ with degeneracy of 3 and the other levels of $\lambda=\pm(\sqrt{3}-1), \pm\sqrt{2}, \pm(\sqrt{3}+1)$ are all single levels. For $n=3$ there are 27 levels (shown by squares) in the spectrum. There is a highest peak at the center level $(\lambda=0)$ with degeneracy of 9. There are two peaks at the levels of $\lambda=\pm\sqrt{2}$ with degeneracy of 3. And the other 12 levels are all single levels. For $n=4$ there are 81 levels (shown by diamonds) in the spectrum. There is a highest peak at the center level $(\lambda=0)$ with degeneracy of 27. There are two peaks at the levels of $\lambda=\pm\sqrt{2}$ with degeneracy of 9. There are four peaks at the levels of $\lambda=\pm\sqrt{2}$ with degeneracy of 3. And the other 24 levels are all single levels and so forth.

In order to study this nature further, we have done numerical calculations for the spectra of the adjacency matrices with the sizes of $3^5=243$, $3^6=729$, $3^7=2187$, and $3^8=6561$, respectively. From this we have confirmed ourselves that this nature is numerically exact at any generation of the network up to $n=8$. We show this in Fig. 3 for the cases of $3^5=243$ $(n=5)$, $6^6=729$ $(n=6)$, and $3^7=2187$ $(n=7)$.

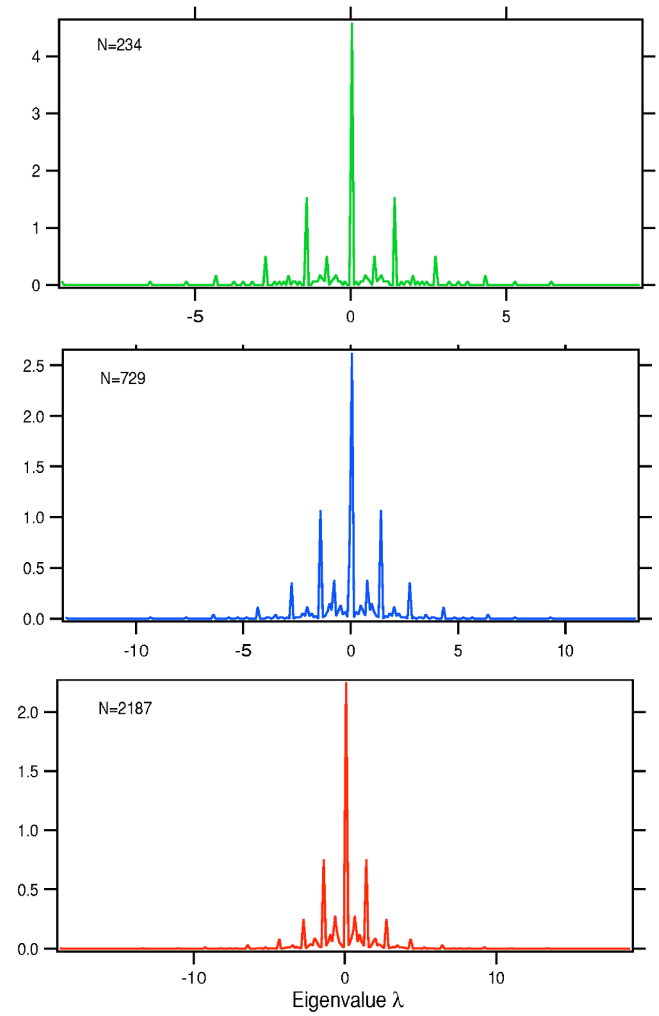

FIG. 3. (Color online) The spectrum of adjacency matrix for the deterministic scale-free network. The spectrum of adjacency matrix for the network of the nth generation is shown for $n=5$, $n=6$, and $n=7$, respectively.

The above nature is remarkable. It enables us to calculate the exact sequence of degeneracies in the spectrum at any generation of the network, apart from finding the exact eigenvalues. By counting the numbers of the levels and its degeneracies, we find the following very important result.

(iv) Let us denote μ_j the degeneracy of the jth peak in the spectrum for the network of the nth generation. Denote by Q_j the number of the eigenvalues having the same degeneracy of μ_j. We find

$$\mu_j = 3^j \quad \text{for } j=0,1,\ldots,n-1, \tag{21}$$

$$Q_j = \begin{cases} 2^{n-1-j} & \text{for } j=1,2,\ldots,n-1, \\ 3\times 2^{n-1} & \text{for } j=0. \end{cases} \tag{22}$$

From this we can check whether or not the above formula is correct. For this purpose, we just reproduce the total number of eigenvalues of the adjacency matrix as follows:

$$\sum_{j=0}^{n-1} \mu_j Q_j = \mu_0 Q_0 + \mu_1 Q_1 + \cdots + \mu_{n-1} Q_{n-1} = 3^n. \quad (23)$$

This is nothing but the total number of eigenvalues in the spectrum. Hence, the formula is proved.

Now we would like to point out the particular nature of the eigenvectors (i.e., the states) corresponding to the eigenvalues: (v) *The eigenvectors (i.e., the states) are very localized*. The states of $\lambda=0$ first appearing in the network of the first generation are localized only on the pairs of leaf nodes such as the pair of 2 and 3, the pair of 5 and 6, the pair of 8 and 9, etc. The states of $\lambda=\pm\sqrt{2}$ first appearing in the network of the first generation are localized only on the smallest triples centered at the root hub such as the triple of 1, 2, and 3, the triple of 4, 5, and 6, the triple of 7, 8, and 9, etc. The states of $\lambda=\pm(\sqrt{3}\pm 1)$ first appearing in the network of the second generation are localized within the subnetworks having the size of that of the second generation, such as the network of 1–9, etc. The states of $\lambda=\pm\sqrt{2(3\pm\sqrt{3}\pm\sqrt{11\pm 6\sqrt{3}})}$ first appearing in the network of the third generation are localized within the subnetworks having the size of that of the second generation, such as the network of 1–27, etc., and so forth. And the number of such subnetworks gives its degeneracy.

Thus, we find that the larger the eigenvalue, the larger the extent of the eigenvector. Hence, we would like to conclude that the eigenvector with the maximum eigenvalue (say, λ_{max}) is most delocalized while the eigenvalues of $\lambda=0$ are extremely localized at the leaf nodes in the network. This nature explains the meaning of the spectrum shown in Figs. 2 and 3. Therefore, since we see similar spectra in most SFN models, we may expect that the same holds true for every SFN. This is a very interesting point in the study of SFN's.

VII. NATURE OF THE EIGENVALUE PROBLEM

A. Simple observations for $D_n(\lambda)$

We are going to treat analytically the above result in the previous section. To do this, let us consider some interesting natures that $D_n(\lambda)$ can be factored in terms of the polynomials of $f_n(\lambda)$, where $f_n(\lambda)$ is an even function of the (2^n)th-order polynomial of λ such that $f_n(\lambda)=f_n(-\lambda)$; this symmetry in the eigenvalues is due to the bipartite nature of the DSFN. For example, the polynomials are given explicitly such as

$$f_1(y) = y - 2,$$

$$f_2(y) = y^2 - 8y + 4,$$

$$f_3(y) = y^4 - 24y^3 + 104y^3 - 96y + 16,$$

$$f_4(y) = y^8 - 64y^7 + 1104y^6 - 742y^5 + 22112y^4 - 29696y^3 + 17664y^2 - 4096y + 256, \quad (24)$$

and so forth, where $y=\lambda^2$. In general, this nature can be summarized as the following conjectures.

Conjecture 1: For the polynomial with even suffix of $n=2m$, it is always factorized as

$$f_{2m}(\lambda) = g_m(\lambda) h_m(\lambda), \quad (25)$$

where $g_m(\lambda)$ and $h_m(\lambda)$ are (2^m)th-order polynomials of λ.

Conjecture 2:

$$D_n(\lambda) = \lambda^{3^{n-1}}\{f_1(\lambda)\}^{3^{n-2}}\{f_2(\lambda)\}^{3^{n-3}}\cdots\{f_{n-2}(\lambda)\}^3 f_{n-1}(\lambda) f_n(\lambda). \quad (26)$$

The meaning of the second conjecture is now very clear. (1) The spectrum is symmetrical around the center level of $\lambda=0$ (see Figs. 2 and 3). This means that if λ is an eigenvalue of the adjacency matrix, then so if for $-\lambda$. (2) The numbers of power exponents in Eq. (26) represent those of the degeneracies of the eigenvalues. For example, the $\lambda=0$ level consists of 3^{n-1} (i.e., the degeneracy of 3^{n-1}) and this corresponds to the highest peak at the center of $\lambda=0$ in the spectrum. The $\lambda=\pm\sqrt{2}$ levels consist of 3^{n-2} (i.e., the degeneracy of 3^{n-2}) and these correspond to the second highest peaks at the center of $\lambda=\pm\sqrt{2}$ in the spectrum and so forth. This proves the formulas of Eqs. (21) and (22), previously found.

The validity of the conjectures is also supported by our numerical calculations as mentioned before. But the exact proofs have not been made yet, however.

B. Roots of the irreducible polynomials

We now consider zeros (i.e., roots) of the irreducible polynomials of $f_n(\lambda)$ for $k=1,2,\ldots,n$. As studied before, the order of $f_n(\lambda)$ is 2^n and it is a function of λ^2. Let us denote $y=\lambda^2$. Then, $f_n(\lambda)$ becomes a function of y such as $f_n(y)$.

Since $f_n(y)$ is now a (2^n)th-order polynomial of y, it has to consist of 2^n zeros. Then the meaning of irreducibility is the following: $f_n(y)$ $(k=1,2,\ldots,n)$ does not share common roots with the other generations of the polynomials. This can be proved by the Sturm theorem for polynomials [32].

From the knowledge of algebra, if there is a series of irreducible polynomials, then roots of the polynomial of order $n-1$ always exist in the intervals between the roots of the polynomial of order n. Therefore, the maximum root of the polynomial of order n exceeds that of order $n-1$. In our problem, the irreducible polynomial $f_n(\lambda)$ is of order 2^n and it gives the newly appearing eigenvalues. Therefore, there are 2^n eigenvalues in the network of the nth generation.

On the other hand, the previously appearing eigenvalues are given as the roots of the irreducible polynomials of order up to $n-1$—i.e., $f_1(\lambda), f_2(\lambda), \ldots, f_{n-1}(\lambda)$. Therefore, there are $1+2+\cdots+2^{n-1}=2^n-1$ eigenvalues already in the spectrum. Thus, the number of newly appearing eigenvalues is exactly one more than that of the previously existing eigenvalues. Hence, by the knowledge of algebra, the eigenvalues of the network of the nth generation sandwich the previously existing eigenvalues such that the maximum eigenvalue may exceed that of the previous generation. As we iterate the network, n becomes large indefinitely. Therefore, the maximum eigenvalue develops further.

To understand the above nature further, we have numerically investigated the growth of the maximum eigenvalue $\lambda_{max}(n)$ at the nth generation up to $n=8$. Define the ratio by $R(n)=\lambda_{max}(n)/\lambda_{max}(n-1)$. The result is as follows: $R(1)=1.932\ldots$, $R(2)=1.583\ldots$, $R(3)=1.486\ldots$, $R(4)=1.447\ldots$, $R(5)=1.430\ldots, \ldots$. From this we see that as n becomes large the ratio tends to the number $\sqrt{2}=1.4142\ldots$. Thus, we are led to the following conjecture.

Conjecture 3: As $n\to\infty$,

$$\lambda_{max} \to 2^{n/2}. \tag{27}$$

This conjecture can be proved from the nature of the series of the irreducible polynomials of Eq. (24). Let us first consider the case of $n=$even. As we have discussed conjecture 1, if $n=$even, then the polynomial can be factorized as $y^{2^{n-1}}-\alpha_1(n)y^{2^{n-1}-1}+\alpha_2(n)y^{2^{n-1}-2}+\cdots=[\lambda^{2^{n-1}}-\beta_1(n)\lambda^{2^{(n-1)}-1}+\cdots][\lambda^{2^{n-1}}+\beta_1(n)\lambda^{2^{(n-1)}-1}+\cdots]=0$, where $\beta_1(n)=2^{n/2}$. Therefore, the maximum eigenvalue is given by $\lambda^{2^{n-1}}-\beta_1(n)\lambda^{2^{(n-1)}-1}-\cdots=0$. Dividing the above polynomial by $\lambda^{2^{(n-1)}-1}$ yields $\lambda=\beta_1(n)+O(1/\lambda)+\cdots$. Since $\lambda_{max}\to\infty$ as $n\to\infty$, we can obtain the approximation of the maximum eigenvalue by a perturbation method. Hence, for $n=$even, we obtain $\lambda_{max}(n)=2^{n/2}+O(2^{(n-1)/2})+\cdots$, such that $\lambda_{max}(n)>2^{n/2}$. Similarly, if $n=$odd, then we can apply the same perturbational argument to the irreducible polynomial; then, we have $y=a_1(n)-O(1/y)+\cdots$, where $a_1(n)=3\times 2^n$ and $y=\lambda^2$. Hence, we obtain

$$\lambda_{max}(n) = \sqrt{3}\times 2^{n/2} - O\left(\frac{1}{\lambda}\right) + \cdots , \tag{28}$$

such that $\lambda_{max}(n)<\sqrt{3}\times 2^{n/2}$. This argument supports the conjecture. Thus, we expect that as $n\to\infty$ $\lambda_{\max}(n)\to 2^{n/2}$.

In this way, the solution of the series of the irreducible polynomials is very crucial in finding the spectrum of the DSFN. This point has been supported by our numerical calculations before.

VIII. HIDDEN SYMMETRY, ZERO MODES, AND INDEX THEOREM

A. Hidden symmetry in the model

Let us consider a particular nature of the DSFN. There is a hidden symmetry in the adjacency matrix A_n. To see this point let us consider the case of $n=2$. As was discussed before, the adjacency matrix in this case is given by Eq. (17) and the eigenvalue problem is

$$\begin{pmatrix} 0&1&1&0&1&1&0&1&1\\ 1&0&0&&&&&&\\ 1&0&0&&&&&&\\ 0&&&0&1&1&&&\\ 1&&&1&0&0&&&\\ 1&&&1&0&0&&&\\ 0&&&&&&0&1&1\\ 1&&&&&&1&0&0\\ 1&&&&&&1&0&0 \end{pmatrix}\begin{pmatrix} x_1\\x_2\\x_3\\x_4\\x_5\\x_6\\x_7\\x_8\\x_9\end{pmatrix} = \lambda\begin{pmatrix} x_1\\x_2\\x_3\\x_4\\x_5\\x_6\\x_7\\x_8\\x_9\end{pmatrix}. \tag{29}$$

This expression depends on the choice of the arrangement of components of the eigenvector $\vec{X}_2=(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9)^t$, where t means its transpose. Reminding ourselves of the bipartite nature of the DSFN, we can rearrange them as follows: $\vec{Y}_2=(x_1,x_4,x_7,x_2,x_5,x_8,x_3,x_6,x_9)^t$. This means that we align the vector-component numbering using mod3. Therefore, we can write it as

$$\tilde{A}_2\begin{pmatrix}\vec{u}\\ \vec{v}\end{pmatrix} = \begin{pmatrix} 0_{3\times 3} & M_2^\dagger \\ M_2 & 0_{6\times 6}\end{pmatrix}\begin{pmatrix}\vec{u}\\ \vec{v}\end{pmatrix} = \lambda\begin{pmatrix}\vec{u}\\ \vec{v}\end{pmatrix}, \tag{30}$$

where $\vec{u}$ and $\vec{v}$ are the three- and six-dimensional vectors, $0_{3\times 3}$ and $0_{6\times 6}$ are zero matrices, and

$$M_2 = \begin{pmatrix} 1&0&0\\ 1&1&0\\ 1&0&1\\ 1&0&0\\ 1&1&0\\ 1&0&1\end{pmatrix}. \tag{31}$$

Here, $\tilde{A}_2=PA_2P^{-1}$ and $\vec{Y}_2=(\vec{u},\vec{v})^t=P\vec{X}_2$, respectively. This is just an interchange of the vector components. Therefore, if there is no confusion between $\tilde{A}_2$ and A_2, we can simply write $\tilde{A}_2$ as A_2. So we identify them as the original adjacency matrix A_2.

From Eq. (30), we find, by simple algebra,

$$M_2^\dagger M_2\vec{u} = \lambda^2\vec{u}, \quad M_2M_2^\dagger\vec{v} = \lambda^2\vec{v}, \tag{32}$$

where $M_2^\dagger M_2$ is the three-dimensional matrix and $M_2M_2^\dagger$ the six-dimensional matrix.

Now, we are able to generalize the above procedure to the adjacency matrix A_n for the DSFN at any generation. In this case Eq. (30) is generalized to

$$A_n\begin{pmatrix}\vec{u}\\ \vec{v}\end{pmatrix} = \begin{pmatrix} 0_{uu} & M_n^\dagger \\ M_n & 0_{vv}\end{pmatrix}\begin{pmatrix}\vec{u}\\ \vec{v}\end{pmatrix} = \lambda\begin{pmatrix}\vec{u}\\ \vec{v}\end{pmatrix}. \tag{33}$$

This then yields

$$M_n^\dagger\vec{v} = \lambda\vec{u}, \quad M_n\vec{u} = \lambda\vec{v}. \tag{34}$$

Here $\vec{u}$ and $\vec{v}$ are the u- and v-dimensional vectors, 0_{uu} and 0_{vv} the $u\times u$ and $v\times v$ zero matrices, and $M_n^\dagger$ and M_n the $u\times v$ and $v\times u$ matrices, respectively, where $u=3^{n-1}$ and v

$=3^n-3^{n-1}$. Therefore, in the same way we have the following:

$$D_n(\lambda) = |\det[M_n^\dagger M_n - \lambda^2 I_n]|^{1/2} |\det[M_n M_n^\dagger - \lambda^2 I_n]|^{1/2} = [\widetilde{D}_{3^{n-1}}(y)\widetilde{D}_{3^n-3^{n-1}}(y)]^{1/2}, \quad (35)$$

where $\widetilde{D}_{3^{n-1}}(y)$ and $\widetilde{D}_{3^n-3^{n-1}}(y)$ $[=y^{3^{n-1}}\widetilde{D}_{3^{n-1}}(y)]$ are factorized in terms of the irreducible polynomials $f_n(\lambda)$. Therefore, Eq. (35) can be reduced to the form of Eq. (26). Thus, this approach can provide us a hint to prove conjecture 2. However, a rigorous proof is out of scope of this paper since it is numerically supported as in Sec. VI.

We would like to mention that similar approaches have been applied to amorphous systems [33,34], the topological localization problem [35], fermion number fractionalization [36], and the Hubbard model [37].

B. Zero modes and index theorem in the deterministic scale-free network

Let us consider *zero modes* (i.e., the eigenvectors having $\lambda=0$) in the spectrum. The zero modes are given by substituting $\lambda=0$ into Eq. (34) such that

$$M_n^\dagger \vec{v} = 0, \quad M_n \vec{u} = 0. \quad (36)$$

The number of zero modes given by $M_n\vec{u}=0$ (or, equivalently, the number of zero modes given by the matrix $M_n^\dagger M_n$) is the dimension of the null space of M_n. This is sometimes called the dimension of the kernel of M_n, which is simply written as $\dim[\ker M_n]$. And the number of zero modes given by $M_n^\dagger\vec{v}=0$ (or, equivalently, the number of zero modes given by the matrix $M_n M_n^\dagger$) is the dimension of the null space of $M_n^\dagger$. This is sometimes called the dimension of the kernel of $M_n^\dagger$, simply written as $\dim[\ker M_n^\dagger]=\dim[\mathrm{coker}\, M_n]$ [38]. Therefore, in the above example of $n=2$, $\dim[\ker M_2]=3$ and $\dim[\ker M_2^\dagger]=0$. This would give a relation $\mathrm{Ind}(M_2) \equiv$ the number of zero modes $=\dim[\ker M_2]-\dim[\ker M_2^\dagger]=3$. This quantity $\mathrm{Ind}(M_n)$ is called the *index* of M_n. And the relation where the number of zero modes coincides with the difference between $\dim[\ker M_n]$ and $\dim[\ker M_n^\dagger]$ is called the *index theorem* [38].

This can be generalized to the adjacency matrix A_n for the DSFN in the nth generation. We now have the index theorem for the DSFN:

$$\mathrm{Ind}(M_n) = \dim[\ker M_n] - \dim[\ker M_n^\dagger] = 3^{n-1}. \quad (37)$$

As we have discussed before, this number is just the number of $\lambda=0$ states, localized at the smallest leaf nodes.

The proof of Eq. (37) is quite simple. For the DSFN, the dimension of the matrix $M_n^\dagger M_n$ is 3^{n-1}, which is the number of hub nodes. And the dimension of the matrix $M_n M_n^\dagger$ is 3^n-3^{n-1}, which is the number of rim nodes. Then, we always find that there are no zero modes in $M_n^\dagger M_n$ in our DSFN. That is, $\dim[\ker M_n^\dagger]=0$. Therefore, the null space of $M_n M_n^\dagger$ is given by the difference between the dimension of the matrix $M_n M_n^\dagger$ and that of the matrix $M_n^\dagger M_n$. This is $3^n-3^{n-1}-3^{n-1}=3^{n-1}$. Hence, $\dim[\ker M_n]=3^{n-1}$.

IX. NATURE OF THE MAXIMUM EIGENVALUE

Let us consider the nature of the maximum eigenvalue. As we studied before, the matrix $M_n^\dagger M_n$ absolutely determines the spectrum of $\lambda>0$, while the matrix $M_n M_n^\dagger$ determines the zero modes of $\lambda=0$ as well as the spectrum. Therefore, in order to determine the spectrum we need study the matrix $M_n^\dagger M_n$.

Let us see a few example of these matrices. The matrix for $n=2$ is obtained from Eq. (31). The matrix for $n=3$ is given as

$$M_3^\dagger M_3 = \begin{pmatrix} 14 & 2 & 2 & 4 & 2 & 2 & 4 & 2 & 2 \\ 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 6 & 2 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 6 & 2 & 2 \\ 2 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 \end{pmatrix}. \quad (38)$$

Here, we find that the diagonal elements of the matrix $M_3^\dagger M_3$ are just the maximum degrees of nodes (i.e., the numbers of the most connected links) in the networks up to the third generation. The maximum diagonal element is the maximum degree of nodes, which is the degree of the root. Hence, it is 14 for $n=3$ (it is 6 for $n=2$).

We find that this is always true for the network of the nth generation. We find that the diagonal elements of the matrix $M_n^\dagger M_n$ are just the maximum degrees of nodes for the networks up to the nth generation. Hence the maximum diagonal element is the maximum degrees of node, which is the degree of the root. It is $k_{\max}=2(2^n-1)$ from Eq. (3).

Let us first derive the lower bound of the maximum eigenvalue, $\lambda_{\max}$. In mathematics, we have the following theorem [39].

Theorem 1: Denote by H a non-negative definite matrix. Denote by $\vec{v}$ a positive definite symmetric matrix. Define a matrix A by $A=H+V$. Suppose that the eigenequations $H|\psi_i\rangle=h_i|\psi_i\rangle$ and $A|\psi_i\rangle=a_i|\psi_i\rangle$ $(i=1,2,\ldots,l)$ are known such that the eigenvalues satisfy $h_1\geqslant h_2\geqslant\cdots\geqslant h_l$ and $a_1\geqslant a_2\geqslant\cdots\geqslant a_l$. And assume that $\langle\psi_i|V|\psi_i\rangle\geqslant 0$. Then, the following holds true:

$$h_s \leqslant a_s \quad \text{for } s=1,2,\ldots,l. \quad (39)$$

We omit the proof here, since it is given in the literature [39].

Let us apply this theorem to the matrix $M_n^\dagger M_n$. We now denote the matrix $M_n^\dagger M_n$ by A in the theorem. Denote by H the matrix having the diagonal matrix components of $M_n^\dagger M_n$ and denote by V the matrix having the off-diagonal matrix components of $M_n^\dagger M_n$. Therefore, $M_n^\dagger M_n$ satisfies the conditions for A in the theorem. The eigenvalue of the matrix A is now $a_i=\lambda_i^2$ and $l=3^{n-1}$. Therefore, we can prove the following theorem for the DSFN.

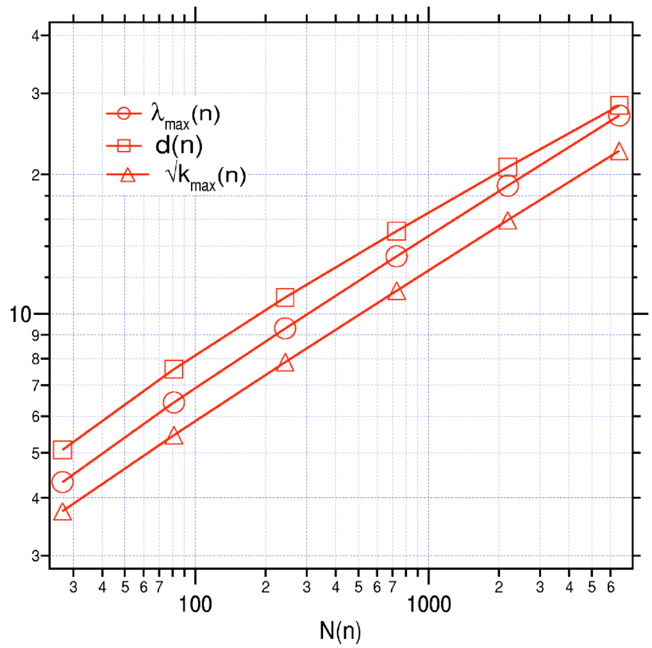

FIG. 4. (Color online) The growth of the maximum eigenvalue λ_{max} and its lower and upper bounds in the DSFN. Here λ_{max} (circles), the lower (triangles), and upper (squares) bounds are shown for the DSFN's of the nth generations up to $n=8$, respectively.

Theorem 2:

$$h_s \leqslant \lambda_s^2 \quad \text{for } s=1,2,\ldots,3^{n-1}. \tag{40}$$

Since the maximum eigenvalue of H is now k_{max}, we have $h_1=k_{max}$. Thus, we finally end up with the following theorem for the DSFN.

Theorem 3:

$$\sqrt{k_{max}} = \sqrt{2^{n+1}-2} \leqslant \lambda_{max}. \tag{41}$$

We have also investigated this theorem numerically in Fig. 4. It shows that the theorem is valid for the DSFN's of the generations up to $n=8$. Thus, this supports the theorem.

Let us next derive the upper bound of the maximum eigenvalue λ_{max}. To do so, let us first consider a particular property of the matrix M_n. As is seen from the matrix such as Eq. (31), the ith column vector of M_n is a vector whose components are 0 or 1 such that the total number of 1 in the column counts the order k_i of the ith hub node. Denote this vector by $\vec{k}_i$. Then we find $\vec{k}_i \cdot \vec{k}_i = k_i$, where by definition $k_1 > k_2 \geqslant \cdots \geqslant k_{3^n}$, since k_1 is the order of the root (i.e., the most connected hub). Then, the matrix can be represented as $M_n=(\vec{k}_1,\vec{k}_4,\ldots,\vec{k}_{3^n-2})$, where the suffices run over all the hub (i.e., root) nodes. Now, we can represent $M_n^\dagger M_n$ in general in the following:

$$M_n^\dagger M_n = \begin{pmatrix} \vec{k}_1\cdot\vec{k}_1 & \vec{k}_1\cdot\vec{k}_4 & \cdots & \vec{k}_1\cdot\vec{k}_{3^n-2} \\ \vec{k}_1\cdot\vec{k}_4 & \vec{k}_4\cdot\vec{k}_4 & \cdots & \vec{k}_4\cdot\vec{k}_{3^n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \vec{k}_1\cdot\vec{k}_{3^n-2} & \vec{k}_2\cdot\vec{k}_{3^n-2} & \cdots & \vec{k}_{3^n-2}\cdot\vec{k}_{3^n-2} \end{pmatrix}, \tag{42}$$

which is a $3^{n-1}\times 3^{n-1}$ symmetric matrix. Therefore, the trace of this matrix is the total number of links such that $\mathrm{tr}(M_n^\dagger M_n)=\Sigma_{i\in root}\vec{k}_i\cdot\vec{k}_i=\Sigma_{i\in root}k_i=\Sigma_{k\in root}kP(k)=L(n)$.

Let us now apply the Perron-Frobenius theorem to our problem. The Perron-Frobenius theorem in linear algebra [31,39] is described as follows.

Theorem 4: Suppose that an $n\times n$ symmetric matrix A has all non-negative entries $a_{ij}\geqslant 0$. Then this satisfies an eigenequation $A|\psi_i\rangle = a_i|\psi_i\rangle$. For any positive constants $c_1,c_2,\ldots,c_n$, the maximum eigenvalue $a_{max}(A)$ satisfies

$$a_{max}(A) \leqslant \max_{1\leqslant i\leqslant n}\left\{\sum_{j=1}^{n}\frac{c_j a_{ij}}{c_i}\right\}. \tag{43}$$

We omit the proof here since this is very well known [31,39].

Regarding A as $M_n^\dagger M_n$ such that $a_{ij}=\vec{k}_i\cdot\vec{k}_j$ and $a_i=\lambda_i^2$ and applying to the Perron-Frobenius theorem, we can prove the following theorem for the DSFN.

Theorem 5:

$$\lambda_{max}^2(M_n^\dagger M_n) \leqslant \sum_{j\in root}\vec{k}_1\cdot\vec{k}_j \equiv b. \tag{44}$$

Since $\Sigma_{j\in root}\vec{k}_1\cdot\vec{k}_j \leqslant \Sigma_{j\in root}\vec{k}_j\cdot\vec{k}_j = L(n)$ together with Eq. (2), we end up with the following theorem for the DSFN.

Theorem 6:

$$\lambda_{max}(M_n^\dagger M_n) \leqslant \sqrt{L(n)} = \sqrt{2(3^n-2^n)}. \tag{45}$$

From theorems 3 and 5, we finally derive the following theorem for the DSFN.

Theorem 7: The maximum eigenvalue of the adjacency matrix of the DSFN is bounded as

$$a \leqslant \lambda_{max} \leqslant b \leqslant c, \tag{46}$$

where $a=\sqrt{2(2^n-1)}$, $c=\sqrt{2(3^n-2^n)}$, and $b=\Sigma_{j\in root}\vec{k}_1\cdot\vec{k}_j$. Although there is no name on the quantity b, this is a much better bound than c, the square root of the total number of links. We note that the above theorem is consistent with conjecture 3 that asserts $\lambda_{max}\to 2^{n/2}$ as $n\to\infty$.

We finally make a comment on the theorem of Chung, Lu, and Vu [31]. As introduced in Introduction, since the DSFN has the exponent of $\gamma=\ln 3/\ln 2<2.5$, this system may belong to the (C2) case. Therefore, if we apply their theorem to our system of the DSFN, then it leads us to the following:

$$\sqrt{k_{max}} \leqslant \lambda_{max} \leqslant \tilde{d}, \tag{47}$$

where $\tilde{d}\propto(4/3)^n=(1.33\ldots)^n$ from Eq. (14). On the other hand, our theorem provides us the lower bound $a\propto 2^{n/2}=(1.414\ldots)^n$ and the upper bound $c\propto 3^{n/2}=(1.732\ldots)^n$. From this, we find that the theorem of Chung, Lu, and Vu [31] holds true until about $n=12$. However, beyond $n=12$ the role

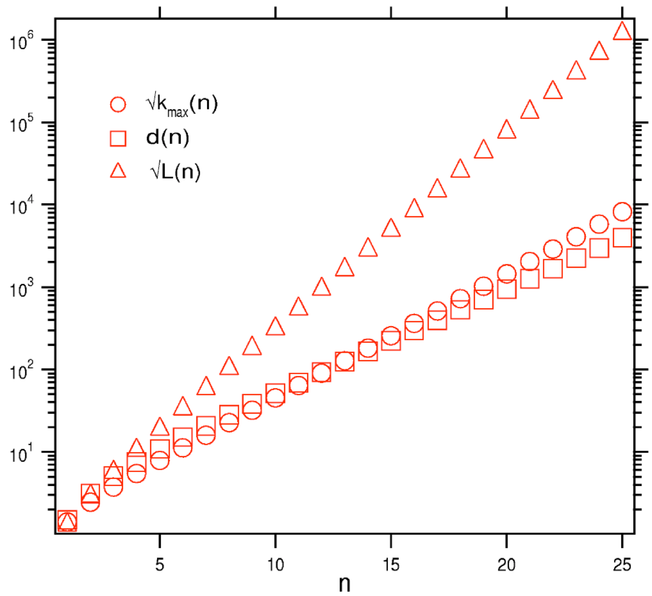

FIG. 5. (Color online) The growth of the square root of the total number of links, $\sqrt{L(n)}$, the maximum order of nodes, $\sqrt{k_{max}}$, and the maximum degree $\tilde{d}(n)$. Clearly, we see that $\sqrt{k_{max}}$ is smaller than $\tilde{d}(n)$ but very close to it under $n=12$. However, $\sqrt{k_{max}}$ exceeds $\tilde{d}(n)$ about $n=12$.

between $\tilde{d}$ and $\sqrt{k_{max}}$ is switched. So their theorem can be violated above $n=12$, although, strictly speaking, their theorem should be applied to the case of $2<\gamma<2.5$. We show this behavior using the above exact expressions in Fig. 5. As shown in Fig. 4, the numerical value of λ_{max} is sandwiched between the upper bound of $\tilde{d}$ and the lower bound of $\sqrt{k_{max}}$. Therefore, the theorem of Chung, Lu, and Vu follows within our calculation up to $n=8$. Unfortunately, a numerical value of λ_{max} is not available above $n=8$ because of the ability of our computer power. So we cannot say anything about where it is located above $n=8$, so far. We may expect that it is still sandwiched between them although the role is switched.

In this way, our rigorous approach provides a concrete example for investigating the validity of the theorem of Chung, Lu, and Vu [31]. This is an advantage of our theory.

X. CONCLUSIONS

In conclusion, we have intensively studied the DSFN that was first studied by Barabási, Ravasz, and Vicsek [28]. We have first studied the geometry of the network and presented the exact numbers of nodes and degrees. Second, we have analytically calculated the exact number of the second-order average degree $\tilde{d}$ [see Eq. (31)], using the numbers of the nodes and degrees. Third, we have numerically calculated the spectra of the adjacency matrix A_n for the network up to the $(n=7)$th generation. Fourth, we have discussed the nature of the adjacency matrix for the network. Fifth, we have shown that there is a hidden symmetry in the adjacency matrix. Finally, we have investigated the maximum eigenvalue λ_{max} in the DSFN. We have shown that λ_{max} is bounded by the lower and upper bounds that are given analytically (see theorem 7). Thus, we believe that our theory presented in this paper gives the first rigorous example in the SFN theory, where most of all quantities in the network theory are analytically obtained. In this context, we would like to call the DSFN the exactly solvable SFN model.

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[1] M. Faloutsos, P. Faloutsos, and C. Faloutsos, Comput. Commun. Rev. **29**, 251 (1999).
[2] Albert-László Barabási, *Linked* (Penguin Books, London, 2002); R. Albert and A.-L. Barabási, Rev. Mod. Phys. **74**, 47 (2002); A.-L. Barabási and E. Bonabeau, Sci. Am. (Int. Ed.) (May), 60 (2003), and references therein.
[3] R. Albert and A.-L. Barabási, Science **286**, 509 (1999); Phys. Rev. Lett. **85**, 5234 (2000); G. Bianconi and A.-L. Barabási, *ibid.* **86**, 5632 (2001).
[4] P. L. Krapivsky, S. Redner, and F. Leyvraz, Phys. Rev. Lett. **85**, 4629 (2000).
[5] P. L. Krapivsky, G. J. Rodgers, and S. Redner, Phys. Rev. Lett. **86**, 5401 (2001).
[6] S. N. Dorogovtsev, J. F. F. Mendes, and A. N. Samukhin, Phys. Rev. Lett. **85**, 4633 (2000).
[7] S. N. Dorogovtsev and J. F. F. Mendes, Europhys. Lett. **52**, 33 (2000).
[8] L. Kullmann and J. Kertész, Phys. Rev. E **63**, 051112 (2001).
[9] S. Bornholdt and H. Ebel, Phys. Rev. E **64**, 035104 (2001).
[10] Z. Burda, J. D. Correia, and A. Krzywicki, Phys. Rev. E **64**, 046118 (2001).
[11] P. Erdös and A. Rényi, Publ. Math. (Debrecen) **6**, 290 (1959); Publ. Math. Inst. Hung. Acad. Sci. **5**, 17 (1960); Acta Math. Acad. Sci. Hung. **12**, 261 (1961); for Erdös and Rényi theory, cf. B. Bollobás, *Random Graphs* (Academic Press, London, 1985).
[12] J. Kleinberg, Nature (London) **406**, 845 (2000).
[13] D. J. Watts and S. H. Strogatz, Nature (London) **393**, 440 (1998).
[14] M. E. J. Newman, C. Moore, and D. J. Watts, Phys. Rev. Lett. **84**, 3201 (2000).
[15] D. S. Callaway, M. E. J. Newman, S. H. Strogatz, and D. J. Watts, Phys. Rev. Lett. **85**, 5468 (2000).
[16] M. E. J. Newman, S. H. Strogatz, and D. J. Watts, Phys. Rev. E **64**, 026118 (2001).
[17] M. Barthélémy and L. A. N. Amaral, Phys. Rev. Lett. **82**, 3180 (1999); **82**, 5180(E) (1999).
[18] L. A. N. Amaral, A. Scala, M. Barthélémy, and H. E. Stanley, Proc. Natl. Acad. Sci. U.S.A. **97**, 11149 (2000).
[19] L. F. Lago-Fernández, R. Huerta, F. Corbacho, and J. A.

Sigüenza, Phys. Rev. Lett. **84**, 2758 (2000).
[20] S. H. Strogatz, Nature (London) **410**, 268 (2001).
[21] S. Lawrence and C. L. Giles, Science **280**, 98 (1998).
[22] B. A. Huberman, P. L. Pirolli, J. E. Pitkov, and Rajan M. Lukose, Science **280**, 130 (1998).
[23] R. Albert, H. Jeong, and A.-L. Barabási, Nature (London) **401**, 130 (1999).
[24] R. Albert, H. Jeong, and A.-L. Barabási, Nature (London) **406**, 378 (2000).
[25] I. J. Farkas, I. Derényi, A.-L. Barabási, and T. Vicsek, Phys. Rev. E **64**, 026704 (2001).
[26] K.-I. Goh, B. Kahng, and D. Kim, Phys. Rev. E **64**, 051903 (2001).
[27] S. N. Dorogovtsev, A. V. Goltsev, J. F. F. Mendes, and A. N. Samukhin, Phys. Rev. E **68**, 046109 (2003).
[28] A.-L. Barabási, E. Ravasz, and T. Vicsek, Physica A **229**, 559 (2001).
[29] E. Ravasz and A.-L. Barabási, Phys. Rev. E **67**, 026112 (2003).
[30] We would like to note that our definition of the exponent γ differs by 1 from the one defined originally by Barabási, Ravasz, and Vicsek of Ref. [30]. This is just a convention of definition for the exponent γ.
[31] F. Chung, L. Lu, and V. Vu, Ann. Comb. **7**, 21 (2003); Proc. Natl. Acad. Sci. U.S.A. **100**, 6313 (2003).
[32] For example, see S. Lang, *Algebra*, 3rd ed. (Addison-Wesley, New York, 1993), p. 454.
[33] D. Weaire and M. Thorpe, Phys. Rev. B **4**, 2508 (1971); M. Thorpe and D. Weaire, *ibid.* **4**, 3518 (1971); M. Thorpe, D. Weaire, and R. Alben, *ibid.* **7**, 3777 (1973).
[34] S. Kirkpatrick and T. Eggarter, Phys. Rev. B **6**, 3598 (1972).
[35] B. Sutherland, Phys. Rev. B **34**, 5208 (1986).
[36] A. J. Niemi and G. W. Semenoff, Phys. Rep. **135**, 99 (1986).
[37] E. H. Lieb, Phys. Rev. Lett. **62**, 1201 (1989).
[38] For example, see M. Nakahara, *Geometry, Topology and Physics* (Adam Hilger, New York, 1990), p. 406.
[39] J. M. Ortega, *Matrix Theory: A Second Course* (Plenum Press, New York, 1987).